

Existence of Hyperbolic Calorons

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Abstract

Recent work of Harland shows that the $SO(3)$ -symmetric, dimensionally-reduced, charge- N self-dual Yang–Mills calorons on the hyperbolic space $\mathbb{H}^3 \times S^1$ may be obtained through constructing N -vortex solutions of an Abelian Higgs model as in the study of Witten on multiple instantons. In this paper we establish the existence of such minimal action charge- N calorons by constructing arbitrarily prescribed N -vortex solutions of the Witten type equations.

1 Introduction

Instantons [1] are topological solitons of the zero-temperature Yang–Mills equations in the Euclidean space \mathbb{R}^4 obtained from the $(3+1)$ -dimensional Minkowski spacetime

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so that the time axis is made imaginary by a Wick rotation. These classical solutions give leading-order contributions in the partition function calculation and describe tunneling between various ground states in quantum field theory formalism [11]. At finite temperature, $T > 0$, one needs to compactify the Euclidean time, t , which means that instantons become time-periodic [8] so that t is confined within the temporal cell

$$0 \leq t \leq \beta = \frac{1}{kT}, \quad (1.1)$$

where k is the Boltzmann constant, so that the normalized partition function assumes an apparently asymmetric form

$$Z = \int \mathcal{D}A \exp \left(- \int_0^\beta \int_{\mathbb{R}^3} L(A) dx dt \right), \quad (1.2)$$

where $L(A)$ is the action density of the gauge field A and $\mathcal{D}A$ denotes the path-integral measure. As in the situation of zero-temperature instantons, finite-action t -periodic gauge field solutions can be stratified [8] by homotopy classes defined by maps from $S^2 \times S^1$ into S^3 and such finite-temperature instantons have been explicitly constructed by Harrington and Shepard [9], and called calorons, which approach the zero-temperature instantons in the limit $\beta \rightarrow \infty$ so that the asymmetry between the spatial and temporal coordinates, x and t , in the partition function (1.2) disappears. Motivated by the work on hyperbolic monopoles [3, 5, 15] in the extreme curvature limit [16] in connection to the Euclidean monopoles, Harland [7] carried out a study of hyperbolic calorons and showed that, within Witten's $SO(3)$ -symmetric dimensionally reduced ansatz [25], hyperbolic calorons may be obtained through constructing multiple vortex solutions of an Abelian Higgs Bogomol'nyi system over a cylindrical stripe. Specifically, a unit charge caloron is presented and large β period and large hyperbolic space curvature limits are discussed [7]. However, the existence of a general charge caloron solution remains unsolved.

Technically, the difficulty lies in the fact that the reduced governing equation is defined over an infinite stripe domain in \mathbb{R}^2 equipped with an exponentially curved metric. As in the study of Witten [25], the vortex equation may be reduced to a Liouville equation which is known to be integrable. However, the sign of the nonlinearity only allows local solvability of the equation [19] and periodicity also introduces additional complexity [17]. The purpose of this paper is to apply the method of nonlinear functional analysis to establish the existence of arbitrarily prescribed multiple vortex solutions of the Abelian Higgs Bogomol'nyi system derived in the work of Harland [7]. Thus it follows that the existence of an arbitrary-charge hyperbolic caloron [7] is obtained.

In the next section, we follow [7] to introduce the hyperbolic caloron problem. In particular, we recall the Bogomol'nyi equations of Harland [7] defined over a cylindrical stripe, similar to Witten's equations for the dimensionally reduced $SO(3)$ -symmetric instanton problem [25]. In Section 3, we recall the governing elliptic equation in terms of the coordinate variables. In Section 4, we prove the existence of

solutions by using a variational approach and a sub- and supersolution argument similar to the method used in constructing the Witten type instanton solutions in $4m$ dimensions [20, 21] systematically developed by Tchrakian [22, 23, 24]. Note that, unlike the problem in [20, 21], the exponential decay of the curved metric and finite periodicity make it hard to gain precise information of a solution at infinity. Although we know that the solution remains bounded, we do not know whether it has a definite asymptotic value at infinity. In particular, we do not know its uniqueness. In Section 5, we deduce some suitable decay estimates near the boundaries of the stripe domain for the solution obtained which allow us to compute the associated topological charge realized as the total magnetic flux, or the second Chern class, explicitly. In Section 6, we obtain decay estimates for the gradient of the solution obtained. These estimates make the formal Bogomol'nyi reduction legitimate and lead us to the conclusion in Section 7 that the action of a charge N caloron represented by the gauge field A carries the anticipated minimum action, $S(A) = 2\pi^2 N$, in normalized units.

2 Hyperbolic calorons and vortices

Following Harland [7], the radial coordinate R of a point (x^1, x^2, x^3) in the hyperbolic ball \mathbb{H}^3 is given by $R^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ which is confined in the interval $0 \leq R < S$ with $S > 0$ the scalar curvature of \mathbb{H}^3 . The temporal coordinate $x^0 = t$ is of period $\beta > 0$ and parametrizes S^1 . Using $d\Omega^2$ to denote the metric on the standard 2-sphere and introducing the new variable

$$r = \frac{S}{2} \tanh^{-1} \left(\frac{R}{S} \right), \quad 0 \leq r < \infty, \quad (2.1)$$

the metric for $\mathbb{H}^3 \times S^1$ is given by

$$ds^2 = dt^2 + dr^2 + \Xi^2 d\Omega^2, \quad (2.2)$$

where the conformal factor $\Xi > 0$ is defined by the function

$$\Xi = \Xi(r) = \frac{S}{2} \sinh \left(\frac{2r}{S} \right), \quad r > 0. \quad (2.3)$$

Let A be an $su(2)$ -valued connection 1-form or the gauge field, $A = A_\mu dx^\mu$, with the associated curvature 2-form $F_A = dA + [A, A]$, and $*$ the Hodge star operator induced from the metric (2.2) over the manifold $M = \mathbb{H}^3 \times S^1$. The Yang–Mills action, $S(A)$, of A , is given in the standard form

$$S(A) = - \int_M \text{Tr}(F_A \wedge *F_A), \quad (2.4)$$

accompanied with the topological charge

$$Q(A) = c_2(A) = -\frac{1}{8\pi^2} \int_M \text{Tr}(F_A \wedge F_A), \quad (2.5)$$

expressed formally as the second Chern class or the first Pontryagin class. Finite-action condition implies [7] that the field decays appropriately at the boundary of M so that one can recognize the topological lower bound

$$S(A) \geq 2\pi^2 |Q(A)|. \quad (2.6)$$

A caloron has a positive charge, $Q(A) > 0$, saturates the lower bound (2.6), $S(A) = 2\pi^2 Q(A)$, and satisfies the self-dual equation

$$F_A = *F_A. \quad (2.7)$$

As in [7], we are interested in the vanishing holonomy situation where $Q(A)$ is an integer. To proceed further, we follow [7, 13] to represent the $SO(3)$ -symmetric gauge field A in terms of an Abelian gauge field $a = a_t dt + a_r dr$ and a complex scalar Higgs field $\phi = \phi_1 - i\phi$ as

$$A = -\frac{1}{2}(qa + \phi_1 dq + [\phi_2 + 1]q dq), \quad (2.8)$$

where $q = x^j \sigma^j / R$ with σ^j ($j = 1, 2, 3$) denoting the Pauli spin matrices. Thus, in terms of the reduced Abelian curvature $F_a = da$ and connection $d_a \phi = d\phi + ia\phi$, the Yang–Mills action over $M = \mathbb{H}^3 \times S^1$ boils down into an Abelian Higgs action over the cylindrical stripe

$$\mathcal{M} = \{(r, t) \mid 0 < r < \infty, 0 \leq t \leq \beta\}, \quad (2.9)$$

equipped with the metric

$$d\ell^2 = \frac{1}{\Xi^2}(dt^2 + dr^2), \quad (2.10)$$

of the form

$$S(A) = S(\phi, a) = \frac{\pi}{2} \int_{\mathcal{M}} (F_a \wedge *F_a + 2d_a \phi \wedge *\overline{d_a \phi} + *(1 - |\phi|^2)^2), \quad (2.11)$$

where now $*$ is understood to be the Hodge dual with respect to the metric (2.10) on \mathcal{M} . Furthermore, the topological charge $Q(A)$ becomes the first Chern number

$$Q(A) = \frac{1}{2\pi} \int_{\mathcal{M}} F_a = c_1(a). \quad (2.12)$$

Accordingly, using the method of Bogomol'nyi [4, 12], it can formally be shown that the lower bound stated in (2.6) is attained if the pair (ϕ, a) satisfies the self-dual vortex equations over \mathcal{M} ,

$$d_a \phi + i * d_a \phi = 0, \quad (2.13)$$

$$*F_a = 1 - |\phi|^2. \quad (2.14)$$

These equations can also be reduced from the original Yang–Mills equation (2.7) via the $SO(3)$ -symmetric ansatz (2.8) as described in [7]. In view of such a connection, our main existence theorem for calorons may be stated as follows.

Theorem 2.1. *For any integer $N \geq 1$, the self-dual Yang–Mills equation (2.7) over the hyperbolic space $\mathbb{H}^3 \times S^1$ has a $2N$ -parameter family of smooth solutions, say $\{A\}$, realizing the prescribed topological invariant $Q(A) = c_2(A) = N$ so that the action (2.4) saturates the topological lower bound stated in (2.6), $S(A) = 2\pi^2 Q(A) = 2\pi^2 N$. In fact, such solutions may be obtained by constructing multivortex solutions of (2.13) and (2.14) representing N vortices realized as zeros of the complex Higgs field ϕ over a cylindrical 2-surface \mathcal{M} defined in (2.9) and equipped with the metric (2.10).*

In the subsequent sections, we aim at solving the coupled equations (2.13) and (2.14), which belong to a category of gauge field equations over Riemann surfaces known as Hitchin's equations [10].

3 Elliptic governing equation

It will be convenient to rewrite (2.13) and (2.14) in terms of the (r, t) -coordinates as in [7]. Thus these equations become

$$D_r \phi + i D_t \phi = 0, \quad (3.1)$$

$$\Xi^2 (\partial_t a_r - \partial_r a_t) = 1 - |\phi|^2, \quad (3.2)$$

where $D_r \phi = \partial_r \phi + i a_r \phi$ and $D_t \phi = \partial_t \phi + i a_t \phi$ are the gauge-covariant derivatives of ϕ and the field configurations are all β -periodic in the variable t .

Use complex variables to represent the equations with the convention

$$z = r + it, \quad a = a_r + i a_t, \quad \partial_z = \partial = \frac{1}{2}(\partial_r - i \partial_t), \quad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_r + i \partial_t). \quad (3.3)$$

Then (3.1) takes the form

$$\bar{\partial} \phi = -\frac{i}{2} a \phi, \quad (3.4)$$

so that, away from the zeros of ϕ , we have $a = 2i \bar{\partial} \ln \phi$. On the other hand, noting that

$$\partial a - \bar{\partial} \bar{a} = i(\partial_r a_t - \partial_t a_r), \quad (3.5)$$

we see that the nontrivial rt -component of the curvature of a may be represented as

$$F_{rt} = (F_a)_{rt} = \partial_r a_t - \partial_t a_r = 2\partial \bar{\partial} \ln |\phi|^2 = \frac{1}{2}(\partial_r^2 + \partial_t^2) \ln |\phi|^2 = \frac{1}{2} \Delta \ln |\phi|^2. \quad (3.6)$$

In view of (3.6), we can rewrite (3.2) away from the zeros of ϕ as

$$\frac{1}{2} \Xi^2 \Delta \ln |\phi|^2 = |\phi|^2 - 1. \quad (3.7)$$

It is well known that the zeros of ϕ are discrete and have integer multiplicities. Let these zeros be

$$p_1, p_2, \dots, p_N, \quad (3.8)$$

where and in the sequel, a zero of multiplicity m appears in the list (3.8) m times. Set $u = \ln |\phi|^2$. Then, over the full space \mathcal{M} , (3.7) becomes

$$\Delta u = \frac{2}{\Xi^2}(e^u - 1) + 4\pi \sum_{j=1}^N \delta_{p_j}. \quad (3.9)$$

We are to look for a solution of (3.9) satisfying the boundary condition

$$u(r, t) \rightarrow 0 \text{ as } r \rightarrow 0; \quad u(r, t) \text{ is of period } \beta \text{ in the variable } t; \quad (3.10)$$

u stays bounded over \mathcal{M} .

Conversely, the Higgs field ϕ has the amplitude $|\phi|^2 = e^u$ in terms of a solution u of (3.9) and the Abelian gauge field a_t, a_r may be constructed from utilizing (3.1) to give us

$$a_t = \text{Re}\{2i\bar{\partial} \ln u\}, \quad a_r = \text{Im}\{2i\bar{\partial} \ln u\}, \quad (3.11)$$

which allows us to find the useful relation

$$|D_t \phi|^2 + |D_r \phi|^2 = \frac{1}{2} e^u |\nabla u|^2. \quad (3.12)$$

Returning to the equation (3.9), staying away from the points p_1, p_2, \dots, p_N , and using the translation

$$u = 2 \ln \Xi + v, \quad (3.13)$$

we see that the function satisfies the Liouville equation

$$\Delta v = 2e^v, \quad (3.14)$$

which is integrable. However, such an integrability is only local because (3.14) is known [19] to have no entire solution over \mathbb{R}^2 , although our problem requires that the solution be of period β in its t variable. In doubly periodic case, the solutions to the Liouville equation are considered by Olesen [17, 18] in the context of nonrelativistic Chern–Simons vortices and electroweak vortices over periodic lattices where one needs to use the elliptic functions [2, 6, 14] of Weierstrass since the holomorphic functions representing solutions of (3.14) are periodic. In our situation here, complication comes from both the periodicity of the solution v of (3.14) in the t variable and unboundedness of v in the r variable as $r \rightarrow 0$ and $r \rightarrow \infty$, respectively, as a consequence of the form of the background function Ξ given in (2.3). Due to these difficulties, we choose to use analytic methods to study (3.9) directly, rather than treating it as an integrable equation, so that the desired boundary conditions and arbitrarily prescribed distribution of vortices, as well as their relations to the calculation of topological charges and minimal actions, can all be realized directly and readily.

4 Construction of solution to the vortex equation

For $x = (r, t)$, set

$$U_0(x) = - \sum_{j=1}^N \ln(1 + |x - p_j|^{-2}), \quad x \in \mathbb{R}^2. \quad (4.1)$$

Then $U_0 < 0$. Let $\eta(x)$ be a smooth cut-off function such that $0 \leq \eta \leq 1$, η is of compact support in the rectangle $(0, \infty) \times (0, \beta)$, and

$$\eta = 1 \quad \text{in a neighborhood of } \{p_1, p_2, \dots, p_N\}. \quad (4.2)$$

Then $u_0 = \eta U_0$ is of compact support in $(0, \infty) \times (0, \beta)$, $u_0 \leq 0$, and

$$\Delta u_0 = 4\pi \sum_{j=1}^N \delta_{p_j} - g(r, t) \quad (4.3)$$

with $g \in C_0^\infty((0, \infty) \times (0, \beta))$ (set of smooth functions of compact supports).

Rewrite u in (3.9) as $u = u_0 + v$. Then we have

$$\Delta v = \frac{2}{\Xi^2} (e^{u_0+v} - 1) + g(r, t) \quad (4.4)$$

which is so defined that we are interested in solution depending on the radial variable $r > 0$ and the Euclidean time variable t of period β .

Lemma 4.1. *The function $v^+ = -u_0$ is an upper solution of (4.4).*

Proof. The function v^+ is clearly of period β in the variable t . Besides, in sense of distribution, we have from (4.3) the inequality

$$\Delta v^+ = \Delta(-u_0) \leq \frac{2}{\Xi^2} (e^{u_0+v^+} - 1) + g(r, t) \quad (4.5)$$

as desired.

We next construct a lower solution of (4.4). For this purpose, we define

$$G(r) = \max\{\max\{g(r, t), 0\} \mid 0 \leq t \leq \beta\}. \quad (4.6)$$

We consider the boundary value problem

$$w_{rr} = \frac{2}{\Xi^2} (e^w - 1) + G(r), \quad r > 0, \quad (4.7)$$

$$w(0) = 0, \quad w(\infty) = w_\infty, \quad (4.8)$$

where $w_\infty \leq 0$ is an undetermined constant.

The singular nature of the equation (4.7) does not allow us to approach it directly. Instead, we consider the approximate boundary value problem

$$w_{rr} = \frac{2}{\Xi^2}(e^w - 1) + G(r), \quad \varepsilon_n < r < K_n, \quad (4.9)$$

$$w(\varepsilon_n) = 0, \quad w(K_n) = 0, \quad (4.10)$$

for $n = 1, 2, \dots$. Here $\{\varepsilon_n\}$ and $\{K_n\}$ are monotone sequences of positive numbers with $\varepsilon_n < K_n$, $\text{supp}(G) \subset (\varepsilon_n, K_n)$, $n = 1, 2, \dots$, and

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \lim_{n \rightarrow \infty} K_n = \infty. \quad (4.11)$$

Lemma 4.2. *The boundary value problem consisting of (4.9) and (4.10) has a unique solution which may be obtained by minimizing the functional*

$$I_n(w) = \int_{\varepsilon_n}^{K_n} \left\{ \frac{1}{2}(w_r)^2 + \frac{2}{\Xi^2}(e^w - 1 - w) + G(r)w \right\} dr \quad (4.12)$$

in the space $W_0^{1,2}(\varepsilon_n, K_n)$.

Proof. Since $\Xi(r) \geq \Xi(\varepsilon_n) > 0$ for $r \in (\varepsilon_n, K_n)$ and $e^w - 1 - w \geq 0$, we can use the Schwarz inequality and the Poincaré inequality to derive easily the coerciveness of the functional I_n , namely, $I_n(w) \geq C_1 \|w\|_{W_0^{1,2}(\varepsilon_n, K_n)}^2 - C_2$ for some constants $C_1, C_2 > 0$. Hence the existence of critical point of I_n in $W_0^{1,2}(\varepsilon_n, K_n)$ follows which solves the boundary value problem (4.9)–(4.10). The uniqueness can be proved using a maximum principle argument in (4.9)–(4.10).

In order to pass to the $n \rightarrow \infty$ limit, we need the following monotonicity results. For notational convenience, we will use f_r and f' interchangeably to denote the derivative of a function f with respect to the radial variable r .

Lemma 4.3. *Let w_n be the unique solution of the boundary value problem (4.9)–(4.10) obtained in Lemma 4.2. Then there hold the monotonicity relation*

$$I_n(w_n) \geq I_{n+1}(w_{n+1}), \quad n \geq 1, \quad (4.13)$$

and the uniform coerciveness lower bound

$$I_n(w_n) \geq C_1 \int_{\varepsilon_n}^{K_n} (w'_n(r))^2 dx - C_2, \quad n \geq 1, \quad (4.14)$$

where $C_1, C_2 > 0$ are independent of n . Furthermore, the sequence $\{w_n\}$ is monotone-ordered according to

$$0 > w_n > w_{n+1}, \quad \varepsilon_n < r < K_n, \quad n \geq 1. \quad (4.15)$$

Proof. First we recall that for $n \geq 1$ the function w_n is the unique minimizer of the functional I_n in $W_0^{1,2}(\varepsilon_n, K_n)$. Set $w_n = 0$ for $r < \varepsilon_n$ and $r > K_n$. Then $w_n \in W_0^{1,2}(\varepsilon_{n+1}, K_{n+1})$ and $I_{n+1}(w_n) = I_n(w_n)$. However, w_{n+1} is the global minimizer of I_{n+1} in $W_0^{1,2}(\varepsilon_{n+1}, K_{n+1})$. Hence $I_{n+1}(w_{n+1}) \leq I_{n+1}(w_n)$ and (4.13) is established.

Let $f(r)$ be a function so that $f(r) = 0$ when $r > 0$ is sufficiently small or large. Then, an integration by parts gives us

$$\int_0^\infty \frac{1}{r^2} f^2(r) dr = 2 \int_0^\infty \frac{1}{r} f(r) f'(r) dr. \quad (4.16)$$

Thus, by the Schwarz inequality, we have

$$\int_0^\infty \frac{1}{r^2} f^2(r) dr \leq 4 \int_0^\infty (f_r)^2 dr. \quad (4.17)$$

Using the elementary inequality $e^w - 1 - w \geq 0$ again and (4.17), we have

$$\begin{aligned} I_n(w_n) &\geq \frac{1}{2} \int_{\varepsilon_n}^{K_n} (w'_n)^2 dr - \left(\int_{\varepsilon_n}^{K_n} r^2 G^2 dr \right)^{1/2} \left(\int_{\varepsilon_n}^{K_n} \frac{w_n^2}{r^2} dr \right)^{1/2} \\ &\geq \frac{1}{4} \int_{\varepsilon_n}^{K_n} (w'_n)^2 dr - 4 \int_{\varepsilon_n}^{K_n} r^2 G^2 dr, \end{aligned} \quad (4.18)$$

which gives us (4.14).

Finally, applying the maximum principle and the condition $G(r) \geq 0$ in (4.9)–(4.10), we see that $w_n < 0$ in (ε_n, K_n) . In particular, $w_{n+1} < 0$ on $[\varepsilon_n, K_n]$. Now in (ε_n, K_n) the function $w_{n+1} - w_n$ satisfies

$$(w_{n+1} - w_n)_{rr} = \frac{2}{\Xi^2} e^{\xi_n} (w_{n+1} - w_n) \text{ where } \xi_n \text{ lies between } w_n \text{ and } w_{n+1}, \quad (4.19)$$

and $(w_{n+1} - w_n)(r) < 0$ for $r = \varepsilon_n$ and $r = K_n$. Applying the maximum principle to (4.19) gives us $w_{n+1} < w_n$ in (ε_n, K_n) or (4.15).

Lemma 4.4. *The sequence $\{w_n\}$ constructed in Lemma 4.3 is weakly convergent in $W_{\text{loc}}^{1,2}(0, \infty)$. The so-obtained weak limit, say w , is a classical solution of the equation (4.7). In fact, the convergence $w_n \rightarrow w$ ($n \rightarrow \infty$) may be achieved in any $C^k[a, b]$ topology for arbitrary $0 < a < b < \infty$. In particular, we have $w \leq 0$ everywhere.*

Proof. From (4.13) and (4.14), we see that there is an absolute constant $C > 0$ such that

$$\sup_n \left\{ \int_{\varepsilon_n}^{K_n} (w'_n(r))^2 dr \right\} \leq C. \quad (4.20)$$

From (4.17) and (4.20), we see that $\{w_n\}$ is bounded in $W^{1,2}(a, b)$ for arbitrary $0 < a < b < \infty$. In view of the monotonicity (4.15), we conclude that $\{w_n\}$ is weakly

convergent in $W^{1,2}(a, b)$. Using extension, we can find a function $w \in W_{\text{loc}}^{1,2}(0, \infty)$ such that $\{w_n\}$ converges to w in $W^{1,2}(a, b)$ for any $0 < a < b < \infty$.

Choose $n_0 \geq 1$ such that $(a, b) \subset (\varepsilon_n, K_n)$ when $n \geq n_0$. Thus, for any test function $\xi \in C_0^1(a, b)$, we have

$$\int_a^b \left\{ w'_n \xi' + \frac{2}{\Xi^2} (e^{w_n} - 1) \xi + G(r) \xi \right\} dr = 0, \quad n \geq n_0. \quad (4.21)$$

Using the weak convergence of $\{w_n\}$ in $W^{1,2}(a, b)$, we see that $\{w_n\}$ is convergent in $C[a, b]$ as well. We can take $n \rightarrow \infty$ in (4.21) to show that w is a weak solution of (4.7) over (a, b) . Since (a, b) is arbitrary, we see that w is a weak solution of (4.7) over the full domain $r > 0$. Standard elliptic theory then implies that w is a classical solution of (4.7).

The convergence in any $C^k[a, b]$ topology for arbitrary $0 < a < b < \infty$ follows from applying elliptic estimates in the equation

$$w''_n = \frac{2}{\Xi^2} (e^{w_n} - 1) + G(r) \quad (4.22)$$

and the property $w_n \rightarrow w$ in $C[a, b]$ as $n \rightarrow \infty$.

Lemma 4.5. *Let w be the solution of (4.7) obtained in Lemma 4.4. Then it satisfies the boundary condition (4.8) for some unique number $w_\infty \leq 0$ so that for arbitrarily small $\varepsilon > 0$ we have $w(r) = O(r^{2-\varepsilon})$ as $r \rightarrow 0$ and $w(r) - w_\infty = O(e^{-4r/S})$ as $r \rightarrow \infty$.*

Proof. Let $\{w_n\}$ be the sequence of solutions of (4.9)–(4.10) obtained in Lemmas 4.2 and 4.3. Then for any $r \in (\varepsilon_n, K_n)$, we have by the Schwarz inequality and (4.20) the uniform bound

$$|w_n(r)| \leq \int_{\varepsilon_n}^r |w'_n(\rho)| d\rho \leq r^{1/2} \left(\int_{\varepsilon_n}^{K_n} (w'_n(r))^2 dr \right)^{1/2} \leq Cr^{1/2}, \quad (4.23)$$

where $C > 0$ is independent of n and r . Hence $w(r) = O(r^{1/2})$ when $r \rightarrow 0$ which is a crude preliminary estimate. To improve it, we consider a comparison function

$$W(r) = Cr^{2-\varepsilon}, \quad r > 0, \quad C > 0, \quad \varepsilon \in (0, 1), \quad (4.24)$$

and set $U = w + W$. Choose $r_0 > 0$ small such that $G(r) = 0$ for $0 < r < r_0$. In view of (4.7), (4.24),

$$\Xi(r) = r \cosh\left(\frac{2\xi(r)}{S}\right), \quad \xi(r) \in (0, r), \quad (4.25)$$

and $w(0) = 0$, we have

$$U_{rr} = \frac{2}{\Xi^2} (e^w - 1) + (2 - \varepsilon)(1 - \varepsilon)r^{-2}W = r^{-2}(K(r)w + (2 - \varepsilon)(1 - \varepsilon)W), \quad (4.26)$$

where $0 < r < r_0$ and $K(r) \rightarrow 2$ as $r \rightarrow 0$. Hence, when r_0 is small, we have $K(r) > (2 - \varepsilon)(1 - \varepsilon)$ for $r \in (0, r_0)$. Inserting this condition into (4.26), we have

$$U_{rr} < r^{-2}K(r)U, \quad 0 < r < r_0. \quad (4.27)$$

Using $U(0) = w(0) + W(0) = 0$ and assuming $C > 0$ in (4.24) is large enough so that $U(r_0) = w(r_0) + W(r_0) > 0$. Applying these in (4.27), we get $U(r) > 0, r \in (0, r_0)$. That is, we have obtained the estimate

$$0 \geq w(r) > -Cr^{2-\varepsilon}, \quad 0 < r < r_0, \quad (4.28)$$

as claimed.

In view of (4.20), we deduce that

$$\int_0^\infty (w_r)^2 dr < \infty. \quad (4.29)$$

Therefore, there is a sequence $\{r_n\}$, $r_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} w_r(r_n) = 0. \quad (4.30)$$

Furthermore, since G in (4.7) is of compact support, there is some $K > 0$ such that $G(r) = 0$ for $r > K$. Thus, using $w \leq 0$ and the definition (2.3), we see that w_{rr} satisfies

$$|w_{rr}(r)| = \frac{2}{\Xi^2} |e^w - 1| \leq \frac{64}{S^2} e^{-\frac{4r}{S}}, \quad r \geq K_0 = \max \left\{ K, \frac{S}{2} \ln 2 \right\}. \quad (4.31)$$

Integrating (4.7), using (4.31), and applying (4.30), we arrive at

$$0 \leq w_r(r) \leq \frac{16}{S} e^{-\frac{4r}{S}}, \quad r > K_0. \quad (4.32)$$

Integrating (4.32), we see that there is some number $w_\infty \leq 0$ such that

$$\lim_{r \rightarrow \infty} w(r) = w_\infty, \quad 0 \leq w_\infty - w(r) \leq 4e^{-\frac{4r}{S}}, \quad r > K_0, \quad (4.33)$$

as anticipated.

To see the uniqueness of w_∞ , we assume there are solutions of (4.7), say W_1 and W_2 such that

$$W_1(\infty) = W_{1,\infty}, \quad W_2(\infty) = W_{2,\infty}, \quad W_{1,\infty} > W_{2,\infty}. \quad (4.34)$$

Let $W = W_1 - W_2$. Then W satisfies

$$W_{rr} = \frac{2}{\Xi^2} e^\xi W, \quad r > 0, \quad (4.35)$$

where ξ lies between W_1 and W_2 . Since $W(0) = 0$ and $W(\infty) > 0$, we have $W(r) > 0$ for all $r > 0$. Otherwise, let $W(r_0) \leq 0$ for some $r_0 > 0$. We may assume that W attains its global minimum at r_0 . If $W(r_0) = 0$, then $W_r(r_0) = 0$, which implies $W \equiv 0$ by the uniqueness of solution to the initial value problem of an ordinary differential equation, contradicting the condition $W(\infty) > 0$. So $W(r_0) < 0$ but this contradicts the fact $W_{rr}(r_0) \geq 0$. Hence $W(r) > 0$ for all $r > 0$.

Since $W(\infty)$ is finite, there is a sequence $\{r_n\}$, $r_n \rightarrow \infty$ as $n \rightarrow \infty$ such that $W_r(r_n) \rightarrow 0$ as $n \rightarrow \infty$. Integrating (4.35) over (r, r_n) and letting $n \rightarrow \infty$, we have

$$W_r(r) = - \int_r^\infty \frac{2}{\Xi^2} e^\xi W \, d\rho, \quad r > 0. \quad (4.36)$$

Using $W(0) = 0$, $W > 0$, and the above, we see that $W(r)$ decreases. In particular, $W(\infty) < 0$, which is another contradiction.

The proof of the lemma is complete.

Despite of the above uniqueness result, we are unable to show that $w_\infty = 0$.

We are now ready to solve (4.4). We can state

Theorem 4.6. *The equation (4.4) has a bounded solution v satisfying $v(r, t) = O(r^{2-\varepsilon})$ as $r \rightarrow 0$ where $\varepsilon > 0$ is arbitrarily small.*

Proof. Let w be the solution of (4.7) stated in Lemma 4.5. Since $u_0 \leq 0$, we have

$$\Delta w \geq \frac{2}{\Xi^2} (e^{u_0+w} - 1) + g(r, t). \quad (4.37)$$

In other words, $v^- = w$ is a lower solution of the equation (4.4). Combining with Lemma 4.1, we have $v^+ \geq 0 \geq v^-$. Using elliptic method, we get a solution v of (4.4) satisfying $v^- \leq v \leq v^+$. Since $v^+ = -u_0$ is of compact support in $(0, \infty) \times (0, \beta)$, we obtain from Lemma 4.5 that v is bounded and satisfies $v(r, t) = O(r^{2-\varepsilon})$ as $r \rightarrow 0$ for any small number $\varepsilon > 0$.

5 Calculation of topological charge

Let $v = v(r, t)$ be the β -periodic solution of (4.4) obtained in Theorem 4.6. Define the β -averaged function by

$$\bar{v}(r) = \frac{1}{\beta} \int_0^\beta v(r, t) \, dt. \quad (5.1)$$

From the uniform decay estimate $v(r, t) = O(r^{2-\varepsilon})$ (when $r > 0$ is small), $\bar{v}(0) = 0$, and the L'Hopital's rule, we have

$$\lim_{r \rightarrow 0} \bar{v}_r(r) = \lim_{r \rightarrow 0} \frac{\bar{v}(r)}{r} = \frac{1}{\beta} \lim_{r \rightarrow 0} \int_0^\beta \frac{1}{r} v(r, t) \, dt = 0. \quad (5.2)$$

On the other hand, since \bar{v} is bounded, there is a sequence $\{r_n\}$, $r_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $\bar{v}_r(r_n) \rightarrow 0$ as $n \rightarrow \infty$. Integrating (4.4) over $(r, r_n) \times (0, \beta)$ and letting $n \rightarrow \infty$, we have

$$\begin{aligned}\bar{v}_r(r) &= \frac{1}{\beta} \int_r^\infty \int_0^\beta \frac{2}{\Xi^2} (1 - e^{v+u_0}) dt d\rho \\ &\leq \int_r^\infty \frac{64}{S^2} e^{-\frac{4\rho}{S}} d\rho \rightarrow 0 \quad \text{as } r \rightarrow \infty,\end{aligned}\tag{5.3}$$

where we have used the property $v + u_0 \leq 0$.

Integrating Δv over $0 < r < \infty, 0 < t < \beta$ and applying (5.2) and (5.3), we have

$$\int_0^\infty \int_0^\beta \Delta v dt dr = \beta \left(\lim_{r \rightarrow \infty} \bar{v}_r(r) - \lim_{r \rightarrow 0} \bar{v}_r(r) \right) = 0.\tag{5.4}$$

On the other hand, recall that $u_0 = \eta U_0$ is of compact support and U_0 may be decomposed as the sum of a regular and singular parts in the form

$$U_0(x) = - \sum_{j=1}^N \ln(1 + |x - p_j|^2) + \sum_{j=1}^N \ln |x - p_j|^2 \equiv \tilde{R}(x) + \tilde{S}(x).\tag{5.5}$$

Thus, with $\mathcal{M} = \{(r, t) \mid 0 < r < \infty, 0 \leq t \leq \beta\}$, we insert (5.5) to have

$$\begin{aligned}\int_{\mathcal{M}} g(r, t) dr dt &= \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M} \setminus \cup_{j=1}^N \{x \mid |x - p_j| \geq \varepsilon\}} (-\Delta u_0) dr dt \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{M} \setminus \cup_{j=1}^N \{x \mid |x - p_j| \geq \varepsilon\}} \Delta(\eta \tilde{S}) dr dt \\ &= \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^N \oint_{|x - p_j| = \varepsilon} \frac{\partial(\eta \tilde{S})}{\partial n} dS_\varepsilon = 4\pi N,\end{aligned}\tag{5.6}$$

where dS_ε is the line element and $\partial/\partial n$ denotes the outnormal derivative on the circle $|x - p_j| = \varepsilon$ ($j = 1, \dots, N$).

Integrating (4.4) over \mathcal{M} and using (5.4) and (5.6), we obtain

$$\int_0^\infty \int_0^\beta \frac{1}{\Xi^2} (1 - e^{v+u_0}) dt dr = 2\pi N.\tag{5.7}$$

This result is important because through the relation $|\phi|^2 = e^u = e^{v+u_0}$ and (3.2), we arrive at the anticipated flux quantization condition

$$\Phi = \int_{\mathcal{M}} F_{tr} dt dr = \int_{\mathcal{M}} \frac{1}{\Xi^2} (1 - |\phi|^2) dt dr = 2\pi N.\tag{5.8}$$

In order to calculate the total action, we need to establish some suitable decay estimates for the gradient of a solution obtained, which will be considered in the next section.

6 Decay estimates for the gradient of solution

In order to compute the action of a multiple instanton, we need to derive suitable decay estimates for the first derivatives of the solution v of (4.4) obtained earlier.

We shall first consider the decay estimates near $r = 0$. Since both $u_0(r, t)$ and $g(r, t)$ are compactly supported in \mathcal{M} , we may choose $r_0 > 0$ sufficiently small so that the supports of u_0 and g are contained in $\{(r, t) \in \mathcal{M} \mid r > 2r_0\}$ (say). Hence v satisfies

$$\Delta v = \frac{2}{\Xi^2}(e^v - 1), \quad 0 < r < 2r_0. \quad (6.1)$$

Lemma 6.1. *Let v be the solution of (4.4) obtained earlier. For any arbitrarily small number $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ independent of r, t such that the estimates*

$$|v_t| \leq C(\varepsilon)r^{2-\varepsilon}, \quad -C(\varepsilon)r^{1-\varepsilon} \leq v_r \leq C(\varepsilon)r^{2-\varepsilon}, \quad 0 < r < r_0, \quad (6.2)$$

hold. In particular, $|\nabla v| \rightarrow 0$ uniformly as $r \rightarrow 0$.

Proof. Since $\Xi(r) = O(r)$, $v(r, t) = O(r^{2-\varepsilon})$ uniformly as $r \rightarrow 0$ for $\varepsilon > 0$ arbitrarily small, and $v \leq 0$, we see that $\Delta v \in L^p(\mathcal{M}_{2r_0})$ for any $p > 2$ (say) where $\mathcal{M}_\delta = \{(r, t) \in \mathcal{M} \mid 0 < r < \delta\}$. Elliptic L^p -estimates indicate that $v \in W^{2,p}(\mathcal{M}_{2r_0})$. Using the embedding $W^{2,p}(\mathcal{M}_{2r_0}) \rightarrow C^1(\overline{\mathcal{M}_{2r_0}})$, we see that $|\nabla v|$ is bounded over \mathcal{M}_{2r_0} .

For any $h > 0$, consider the function

$$v^h(r, t) = \frac{v(r, t+h) - v(r, t)}{h}. \quad (6.3)$$

Then $\{v^h\}$ is uniformly bounded over \mathcal{M}_{r_0} and $v^h(r, t) \rightarrow 0$ as $r \rightarrow 0$. Of course, in view of (6.1), v^h satisfies the equation

$$\Delta v^h = \frac{2}{\Xi^2}e^{u^h}v^h, \quad 0 < r < r_0, \quad (6.4)$$

where $u^h(r, t)$ lies between $v(r, t)$ and $v(r, t+h)$. Using the comparison function W for fixed $\varepsilon > 0$ defined in (4.24) and applying (4.25), we get

$$\begin{aligned} \Delta(v^h + W) &= \frac{2}{\Xi^2}e^{u^h}v^h + (2 - \varepsilon)(1 - \varepsilon)r^{-2}W \\ &\leq r^{-2} \frac{2}{\cosh^2\left(\frac{2\xi(r)}{S}\right)} e^{u^h}(v^h + W), \quad 0 < r < r_0, \end{aligned} \quad (6.5)$$

where we have assumed that $r_0 > 0$ is sufficiently small and applied the uniform limit $v(r, t+h) \rightarrow 0$ as $r \rightarrow 0$. Since $\{v^h\}$ is bounded, we may also assume that $C > 0$ in (4.24) is large enough so that

$$v^h(t, r_0) + W(r_0) \geq 0 \quad \text{for all } t. \quad (6.6)$$

Thus, the boundary condition consisting of $v^h + W = 0$ at $r = 0$ and (6.6), the inequality (6.5), and the maximum principle together lead us to

$$v^h(r, t) + W(r) \geq 0, \quad 0 < r < r_0. \quad (6.7)$$

Similarly, we have

$$\Delta(v^h - W) \geq r^{-2} \frac{2}{\cosh^2\left(\frac{2\xi(r)}{S}\right)} e^{u^h}(v^h - W), \quad 0 < r < r_0, \quad (6.8)$$

and we deduce $v^h - W \leq 0$, $0 < r < r_0$. Summarizing these results, we arrive at $|v^h(r, t)| \leq W(r)$, $0 < r < r_0$. Letting $h \rightarrow 0$, we obtain $|v_t(r, t)| \leq W$, $0 < r < r_0$, as stated in (6.2).

In order to get the decay estimate for v_r as $r \rightarrow 0$, we note that

$$\lim_{r \rightarrow 0} v_r(r, t) = \lim_{r \rightarrow 0} \frac{v(r, t)}{r} = 0, \quad (6.9)$$

by virtue of Theorem 4.6. Differentiating (6.1), we find, using $v \leq 0$, the inequality

$$\frac{2}{\Xi^2} e^v v_r - \frac{4}{\Xi^3} (e^v - 1) \cosh\left(\frac{2r}{S}\right) = \Delta v_r \geq \frac{2}{\Xi^2} e^v v_r, \quad 0 < r < r_0. \quad (6.10)$$

Consequently, we have

$$K(r) v_r \leq r^2 \Delta v_r \leq K(r) v_r + C_1 r^{1-\varepsilon}, \quad 0 < r < r_0, \quad (6.11)$$

where $C_1 > 0$ is an absolute constant and $K = r^2(2/\Xi^2)e^v$ satisfies

$$\lim_{r \rightarrow 0} K(r) = 2. \quad (6.12)$$

Hence, for the function W defined in (4.24), we have

$$r^2 \Delta(v_r - W) \geq K(r)(v_r - W), \quad 0 < r < r_0, \quad (6.13)$$

and

$$\begin{aligned} r^2 \Delta(v_r + W_r) &\leq K(r) v_r + C_1 r^{1-\varepsilon} - \varepsilon(1 - \varepsilon) W_r \\ &\leq K(r) v_r + \frac{C_1}{C} W_r, \quad 0 < r < r_0, \end{aligned} \quad (6.14)$$

where $0 < \varepsilon < 1$ and C is as given in (4.24). We may choose C large enough so that $K(r) \geq C_1/C$, $0 < r < r_0$. Then (6.14) gives us

$$r^2 \Delta(v_r + W_r) \leq K(r)(v_r + W_r), \quad 0 < r < r_0. \quad (6.15)$$

Using the same maximum principle argument in (6.13) and (6.15) as before, we see that, when $C > 0$ in (4.24) is large enough, we have

$$-W_r \leq v_r \leq W, \quad 0 < r < r_0, \quad (6.16)$$

which establishes the decay estimate for v_r stated in (6.2).

We now consider the decay estimate for $|\nabla v|$ as $r \rightarrow \infty$. Since we do not know whether $v \rightarrow 0$ as $r \rightarrow \infty$, we encounter a somewhat delicate situation that v may not lie in $L^2(\mathcal{M})$.

Similar to (6.1), we know that v satisfies

$$\Delta v = \frac{2}{\Xi^2}(e^v - 1), \quad r > \delta, \quad (6.17)$$

where $\delta > 0$ is sufficiently large. For convenience, we set $\mathcal{M}^\delta = \{(r, t) \in \mathcal{M} \mid r > \delta\}$.

Lemma 6.2. *We have $|\nabla v| \in L^2(\mathcal{M}^\delta)$.*

Proof. We may extend v outside \mathcal{M}^δ smoothly to get a new function, say w , so that $w = 0$ for $r < \delta/2$ (say). Hence w satisfies

$$\Delta w = \frac{2}{\Xi^2}(e^w - 1) + h(r, t), \quad (r, t) \in \mathcal{M}, \quad (6.18)$$

where h is of compact support and smooth. Choose a smooth function $\eta(r)$ in $r \geq 0$ such that

$$0 \leq \eta \leq 1; \quad \eta(r) = 1, \quad 0 \leq r \leq 1; \quad \eta(r) = 0, \quad r \geq 2. \quad (6.19)$$

Define $\eta_\rho = \eta(r/\rho)$ for $\rho > 0$. Multiplying (6.18) by $\eta_\rho^2 w$ and integrating, we have

$$\begin{aligned} \int_{\mathcal{M}} \left| \frac{2}{\Xi^2}(e^w - 1) + h \right| \eta_\rho^2 |w| \, dr dt &\geq \int_{\mathcal{M}} \eta_\rho^2 |\nabla w|^2 \, dr dt - 2 \int_{\mathcal{M}} \eta_\rho |\nabla w| |w| |\nabla \eta_\rho| \, dr dt \\ &\geq \frac{1}{2} \int_{\mathcal{M}} \eta_\rho^2 |\nabla w|^2 \, dr dt - 2 \int_{\mathcal{M}} w^2 |\nabla \eta_\rho|^2 \, dr dt \\ &\geq \frac{1}{2} \int_{\mathcal{M}} \eta_\rho^2 |\nabla w|^2 \, dr dt - \frac{C}{\rho}, \end{aligned} \quad (6.20)$$

where $C > 0$ is a constant depending on $|w|_\infty$ and β only. Letting $\rho \rightarrow \infty$ in (6.20) and recalling (6.19), we see that $|\nabla w| \in L^2(\mathcal{M})$ and the lemma follows.

Lemma 6.3. *There is constant $C_\delta > 0$ such that*

$$\sup_{\mathcal{M}^\delta} \{|\nabla v|(r, t)\} \leq C_\delta, \quad (6.21)$$

and $|\nabla v|(r, t) \rightarrow 0$ as $r \rightarrow \infty$.

Proof. Differentiating (6.17), we have

$$\Delta v_t = \frac{2}{\Xi^2} e^v v_t, \quad r > \delta. \quad (6.22)$$

Using Lemma 6.2 and elliptic theory, we see that $v_t \in W^{2,2}(\mathcal{M}^\delta)$. Hence v_t is bounded and $v_t \rightarrow 0$ as $r \rightarrow \infty$.

Similarly, differentiating (6.17) with respect to r , we have

$$\Delta v_r = \frac{2}{\Xi^2} e^v v_r - \frac{4}{\Xi^3} (e^v - 1) \cosh\left(\frac{2r}{S}\right), \quad r > \delta, \quad (6.23)$$

whose right-hand side lies in $L^2(\mathcal{M}^\delta)$. Consequently, v_r is bounded and $v_r \rightarrow 0$ as $r \rightarrow \infty$, which establishes the lemma.

The gradient decay estimates for the solution near the boundary of the domain \mathcal{M} allows us to compute the action in terms of the topological invariant explicitly, which will be carried out in the next section.

7 Calculation of action

Following [7], it will be convenient to express the dimensionally reduced action (2.11) of the gauge field A in terms of the (r, t) -coordinates explicitly as

$$S(A) = \frac{\pi}{2} \int_{\mathcal{M}} \left(\Xi^2 F_{tr}^2 + \left[\frac{1 - |\phi|^2}{\Xi} \right]^2 + 2|D_t \phi|^2 + 2|D_r \phi|^2 \right) dr dt. \quad (7.1)$$

It can be checked that the useful identities

$$|D_t \phi|^2 + |D_r \phi|^2 = i(D_r \phi \overline{D_t \phi} - \overline{D_r \phi} D_t \phi) + |D_r \phi + i D_t \phi|^2, \quad (7.2)$$

$$i(D_t \phi \overline{D_r \phi} - \overline{D_t \phi} D_r \phi) = i(\partial_t [\phi \overline{D_r \phi}] - \partial_r [\phi \overline{D_t \phi}]) - F_{tr} |\phi|^2, \quad (7.3)$$

hold. Inserting (7.2) and (7.3) into (7.1) and applying (3.1) and (3.2), we have

$$\begin{aligned} S(A) &= \frac{\pi}{2} \int_{\mathcal{M}} \left(\left[\Xi F_{tr} - \frac{1}{\Xi} (1 - |\phi|^2) \right]^2 + 2F_{tr} (1 - |\phi|^2) \right. \\ &\quad \left. + 2|D_r \phi + i D_t \phi|^2 - 2i(D_t \phi \overline{D_r \phi} - \overline{D_t \phi} D_r \phi) \right) dr dt \\ &= \pi \int_{\mathcal{M}} F_{tr} dx - i\pi \int_{\mathcal{M}} (\partial_t [\phi \overline{D_r \phi}] - \partial_r [\phi \overline{D_t \phi}]) dr dt. \end{aligned} \quad (7.4)$$

By virtue of (3.12), Lemma 6.1, and Lemma 6.3, we see that the last integral on the right-hand side of (7.4) vanishes. Therefore we obtain the quantized minimum action

$$S(A) = 2\pi^2 N, \quad (7.5)$$

as a consequence of the flux formula (5.8).

In summary, we have seen that our main existence theorem for hyperbolic calorons of arbitrary scalar curvature, time period, and topological charge stated in Section 2 is established in Section 4 through a construction of the solution of the multivortex

equation derived by Harland [7], a computation of the associated topological charge in Section 6, and a calculation of the dimensionally reduced action in Section 7, which is based on the gradient boundary estimates of the solution obtained in Section 6.

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